

ON THE MASS OF THE EXTERIOR BLOW-UP POINTS.

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ABSTRACT. We consider the following problem on open set Ω of \mathbb{R}^2 :

$$\begin{cases} -\Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We assume that :

$$\int_{\Omega} e^{u_i} dy \leq C,$$

and,

$$0 \leq V_i \leq b < +\infty$$

On the other hand, if we assume that V_i s -holderian with $1/2 < s \leq 1$, then, each exterior blow-up point is simple. As application, we have a compactness result for the case when:

$$\int_{\Omega} V_i e^{u_i} dy \leq 40\pi - \epsilon, \quad \epsilon > 0$$

1. INTRODUCTION AND MAIN RESULTS

We set $\Delta = \partial_{11} + \partial_{22}$ on open set Ω of \mathbb{R}^2 with a smooth boundary.

We consider the following problem on $\Omega \subset \mathbb{R}^2$:

$$(P) \begin{cases} -\Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We assume that,

$$\int_{\Omega} e^{u_i} dy \leq C,$$

and,

$$0 \leq V_i \leq b < +\infty$$

The previous equation is called, the Prescribed Scalar Curvature equation, in relation with conformal change of metrics. The function V_i is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of this type were studied by many authors, see [5-8, 10-15]. We can see in [5], different results for the solutions of those type of equations with or without boundaries conditions and, with minimal conditions on V , for example we suppose $V_i \geq 0$ and $V_i \in L^p(\Omega)$ or $V_i e^{u_i} \in L^p(\Omega)$ with $p \in [1, +\infty]$.

Among other results, we can see in [5], the following important Theorem,

Theorem A (Brezis-Merle [5]). *If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the previous problem (P) with, $0 < a \leq V_i \leq b < +\infty$, then, for all compact set K of Ω ,*

$$\sup_K u_i \leq c = c(a, b, m, K, \Omega) \quad \text{if} \quad \inf_{\Omega} u_i \geq m.$$

A simple consequence of this theorem is that, if we assume $u_i = 0$ on $\partial\Omega$ then, the sequence $(u_i)_i$ is locally uniformly bounded. We can find in [5] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of e^{u_i} , precisely, we have in [5]:

Theorem B (Brezis-Merle [5]). If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the previous problem (P) with, $0 \leq V_i \leq b < +\infty$, and,

$$\int_{\Omega} e^{u_i} dy \leq C,$$

then, for all compact set K of Ω ,

$$\sup_K u_i \leq c = c(b, C, K, \Omega).$$

If, we assume V with more regularity, we can have another type of estimates, $\sup + \inf$. It was proved, by Shafrir, see [13], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C \left(\frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

We can see in [7], an explicit value of $C \left(\frac{a}{b} \right) = \sqrt{\frac{a}{b}}$. In his proof, Shafrir has used the Stokes formula and an isoperimetric inequality, see [3]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$, see Brézis-Li-Shafrir [4]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [7]. Also, we can see in [10], an extension of the Brezis-Li-Shafrir to compact Riemann surface without boundary. We can see in [11] explicit form, $(8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses, when the solutions blow-up. Here, the notion of isolated blow-up point is used. Also, we can see in [14] refined estimates near the isolated blow-up points and the bubbling behavior of the blow-up sequences.

We have in [15]:

Theorem C (Wolansky.G.[15]). If (u_i) and (V_i) are two sequences of functions solutions of the problem (P) without the boundary condition, with,

$$0 \leq V_i \leq b < +\infty,$$

$$\|\nabla V_i\|_{L^\infty(\Omega)} \leq C_1,$$

$$\int_{\Omega} e^{u_i} dy \leq C_2,$$

and,

$$\sup_{\partial\Omega} u_i - \inf_{\partial\Omega} u_i \leq C_3,$$

the last condition replace the boundary condition.

We assume that (iii) holds in the theorem 3 of [5], then, in the sense of the distributions:

$$V_i e^{u_i} \rightarrow \sum_{j=0}^m 8\pi \delta_{x_j}.$$

in other words, we have:

$$\alpha_j = 8\pi, \quad j = 0 \dots m,$$

in (iii) of the theorem 3 of [5].

To understand the notations, it is interessant to take a look to a previous prints on arXiv, see [1] and [2].

Our main results are:

Theorem 1. Assume that, V_i is uniformly s -holderian with $1/2 < s \leq 1$, and that :

$$\max_{\Omega} u_i \rightarrow +\infty.$$

Then, each exterior blow-up point is simple.

There are m blow-ups points on the boundary (perhaps the same) such that:

$$\int_{B(x_i^j, \delta_i^j \epsilon')} V_i(x_i^j + \delta_i^j y) e^{u_i} \rightarrow 8\pi.$$

and,

$$\int_{\Omega} V_i e^{u_i} \rightarrow \int_{\Omega} V e^u + \sum_{j=1}^m 8\pi \delta_{x_j}.$$

and,

Theorem 2. Assume that, V_i is uniformly s -holderian with $1/2 < s \leq 1$, and,

$$\int_{B_1(0)} V_i e^{u_i} dy \leq 40\pi - \epsilon, \quad \epsilon > 0,$$

then we have:

$$\sup_{\Omega} u_i \leq c = c(b, C, A, s, \Omega).$$

where A is the holderian constant of V_i .

2. PROOF OF THE RESULT:

Proof of the theorem 1:

Let's consider the following function on the ball of center 0 and radius $1/2$; And let us consider $\epsilon > 0$

$$v_i(y) = u_i(x_i + \delta_i y) + 2 \log \delta_i, \quad y \in B(0, 1/2)$$

This function is solution of the following equation:

$$-\Delta v_i = V_i(x_i + \delta_i y) e^{v_i}, \quad y \in B(0, 1/2)$$

The function v_i satisfy the following inequality (without loss of generality):

$$\sup_{\partial B(0, 1/4)} v_i - \inf_{\partial B(0, 1/4)} v_i \leq C,$$

Let us consider the following functions:

$$\begin{cases} -\Delta v_0^i = 0 & \text{in } B(0, 1/4) \\ v_0^i = u_i(x_i + \delta_i y) & \text{on } \partial B(0, 1/4). \end{cases}$$

By the elliptic estimates we have:

$$v_0^i \in C^2(\bar{B}(0, 1/4)).$$

We can write:

$$-\Delta(v_i - v_0^i) = V_i(x_i + \delta_i y) e^{v_0^i} e^{v_i - v_0^i} = K_1 K_2 e^{v_i - v_0^i},$$

With this notations, we have:

$$\|\nabla(v_i - v_0^i)\|_{L^q(B(0, \epsilon))} \leq C_q.$$

$$v_i - v_0^i \rightarrow G \text{ in } W_0^{1, q},$$

And, because, for $\epsilon > 0$ small enough:

$$\|\nabla G\|_{L^q(B(0,\epsilon))} \leq \epsilon' \ll 1,$$

We have, for $\epsilon > 0$ small enough:

$$\|\nabla(v_i - v_0^i)\|_{L^q(B(0,\epsilon))} \leq 2\epsilon' \ll 1.$$

and,

$$\|\nabla v_i\|_{L^q(B(0,\epsilon))} \leq 3\epsilon' \ll 1.$$

Set,

$$u = v_i - v_0^i, \quad z_1 = 0,$$

Then,

$$-\Delta u = K_1 K_2 e^u, \text{ in } B(0, 1/4),$$

and,

$$\text{osc}(u) = 0.$$

We use Woalnsky's theorem, see [15]. In fact K_2 is a C^1 function uniformly bounded and K_1 is s -holderian with $1/2 < s \leq 1$. Because we take the logarithm in K , the part which contain K_2 have similar proof as in this paper we use the Stokes formula. Only the case of K_1 s -holderian is difficult. For this and without loss of generality, we can assume the $K = K_1 = V_i(x_i + \delta_i y)$. We set:

$$\Delta \tilde{u} = \Delta v_i = \rho = -K e^{\tilde{u}} = -K_1 e^{v_i}$$

Let us consider the following term of Wolansky computations:

$$\int_{B^\epsilon} \text{div}((z - z_1)\rho) \log K + \int_{\partial B^\epsilon} \langle (z - z_1)|\nu \rangle \rho \log K,$$

First, we write:

$$\int_{B^\epsilon} \text{div}((z - z_1)\rho) \log K = 2 \int_{B^\epsilon} \rho \log K + \int_{B^\epsilon} \langle (z - z_1)|\nabla \rho \rangle \log K$$

which we can write as:

$$- \int_{B^\epsilon} \text{div}((z - z_1)\rho) \log K = 2 \int_{B^\epsilon} K \log K e^u + \int_{B^\epsilon} \langle (z - z_1)|\nabla u \rangle K \log K e^u + \int_{B^\epsilon} \langle (z - z_1)|(\nabla K) \log K \rangle e^u,$$

We can write:

$$\nabla(K(\log K) - K) = (\nabla K)(\log K)$$

Thus, and by integration by part we have:

$$\begin{aligned} & \int_{B^\epsilon} \langle (z - z_1)|(\nabla K) \log K \rangle e^u = \int_{B^\epsilon} \langle (z - z_1)|(\nabla(K \log K - K)) \rangle e^u = \\ & = \int_{\partial B^\epsilon} \langle (z - z_1)|\nu \rangle (K \log K - K) e^u - 2 \int_{B^\epsilon} (K \log K - K) e^u - \int_{B^\epsilon} \langle (z - z_1)|\nabla u \rangle (K \log K - K) e^u \end{aligned}$$

Thus,

$$\begin{aligned} & - \left(\int_{B^\epsilon} \text{div}((z - z_1)\rho) \log K + \int_{\partial B^\epsilon} \langle (z - z_1)|\nu \rangle \rho \log K \right) = \\ & = - \int_{\partial B^\epsilon} \langle (z - z_1)|\nu \rangle K e^u + \int_{B^\epsilon} \langle (z - z_1)|\nabla u \rangle K e^u + 2 \int_{B^\epsilon} K e^u \end{aligned}$$

But, we can write the following,

$$\int_{B^\epsilon} \langle (z - z_1) | \nabla u \rangle K e^u = \int_{B^\epsilon} \langle (z - z_1) | \nabla u \rangle (K - K(z_1)) e^u + K(z_1) \int_{B^\epsilon} \langle (z - z_1) | \nabla u \rangle e^u,$$

and, after integration by parts:

$$K(z_1) \int_{B^\epsilon} \langle (z - z_1) | \nabla u \rangle e^u = K(z_1) \int_{\partial B^\epsilon} \langle (z - z_1) | \nu \rangle e^u - 2K(z_1) \int_{B^\epsilon} e^u,$$

Finally, we have, for the Wolansky term:

$$\begin{aligned} & \int_{B^\epsilon} \operatorname{div}((z - z_1)\rho) \log K + \int_{\partial B^\epsilon} (\langle (z - z_1) | \nu \rangle \rho) \log K = \\ &= \int_{B^\epsilon} \langle (z - z_1) | \nabla u \rangle (K - K(z_1)) e^u + \left(2 \int_{B^\epsilon} (K - K(z_1)) e^u \right) + \\ & \quad + \left(\int_{\partial B^\epsilon} \langle (z - z_1) | \nu \rangle (K(z_1) - K) e^u \right) \end{aligned}$$

But, we have soon that if K is s -holderian with $1 \geq s > 1/2$, around each exterior blow-up we have, the following estimate:

$$\begin{aligned} & \int_{B^\epsilon} \langle (z - z_1) | \nabla u \rangle (K - K(z_1)) e^u = \\ &= \int_{B(0, \epsilon)} \langle (y - z_1) | \nabla v_i \rangle (V_i(x_i + \delta_i y) - V_i(x_i)) e^{v_i} dy = \\ &= \int_{B(x_i, \delta_i \epsilon)} \langle (x - x_i) | \nabla u_i \rangle (V_i(x) - V_i(x_i)) e^{u_i} dy = o(1) M_\epsilon \\ &= o(1) \int_{B(x_i, \delta_i \epsilon)} V_i e^{u_i} = o(1) \int_{B^\epsilon} K e^u, \end{aligned}$$

Thus,

$$\int_{B^\epsilon} \operatorname{div}((z - z_1)\rho) \log K + \int_{\partial B^\epsilon} (\langle (z - z_1) | \nu \rangle \rho) \log K = o(1) M_\epsilon = o(1) \int_{B^\epsilon} K e^u$$

We argue by contradiction and we suppose that we have around the exterior blow-up point 2 or 3 blow-up points, for example. We prove, as in a previous paper, that, the last quantity tends to 0. But according to Wolansky paper, see [15]:

$$\int_{B^\epsilon} V_i(x_i + \delta_i y) e^{v_i} \rightarrow 8\pi.$$

Around each exterior blow-up points, there is one blow-up point.

Consider the following quantity:

$$B_i = \int_{B(x_i, \delta_i \epsilon)} \langle (x - x_i) | \nabla u_i \rangle (V_i(x) - V_i(x_i)) e^{u_i} dy.$$

Suppose that, we have $m > 0$ interior blow-up points. Consider the blow-up point t_i^k and the associated set Ω_k defined as the set of the points nearest t_i^k we use step by step triangles which are nearest x_i and we take the mediatrices of those triangles.

$$\Omega_k = \{x \in B(x_i, \delta_i \epsilon), |x - t_i^k| \leq |x - t_i^j|, j \neq k\},$$

we write:

$$B_i = \sum_{k=1}^m \int_{\Omega_k} \langle (x - x_i) | \nabla u_i \rangle (V_i(x) - V_i(x_i)) e^{u_i} dy.$$

We set,

$$B_i^k = \int_{\Omega_k} \langle (x - x_i) | \nabla u_i \rangle (V_i(x) - V_i(x_i)) e^{u_i} dy,$$

We divide this integral in 4 integrals:

$$\begin{aligned}
B_i^k &= \int_{\Omega_k} \langle (x - t_i^k) | \nabla u_i \rangle (V_i(x) - V_i(x_i)) e^{u_i} dy + \int_{\Omega_k} \langle (t_i^k - x_i) | \nabla u_i \rangle (V_i(x) - V_i(x_i)) e^{u_i} dy = \\
&= \int_{\Omega_k} \langle (x - t_i^k) | \nabla u_i \rangle (V_i(x) - V_i(t_i^k)) e^{u_i} dy + \int_{\Omega_k} \langle (x - t_i^k) | \nabla u_i \rangle (V_i(t_i^k) - V_i(x_i)) e^{u_i} dy + \\
&+ \int_{\Omega_k} \langle (t_i^k - x_i) | \nabla u_i \rangle (V_i(x) - V_i(t_i^k)) e^{u_i} dy + \int_{\Omega_k} \langle (t_i^k - x_i) | \nabla u_i \rangle (V_i(t_i^k) - V_i(x_i)) e^{u_i} dy,
\end{aligned}$$

We set:

$$\begin{aligned}
A_1 &= \int_{\Omega_k} \langle (x - t_i^k) | \nabla u_i \rangle (V_i(x) - V_i(t_i^k)) e^{u_i} dy, \\
A_2 &= \int_{\Omega_k} \langle (x - t_i^k) | \nabla u_i \rangle (V_i(t_i^k) - V_i(x_i)) e^{u_i} dy, \\
A_3 &= \int_{\Omega_k} \langle (t_i^k - x_i) | \nabla u_i \rangle (V_i(x) - V_i(t_i^k)) e^{u_i} dy, \\
A_4 &= \int_{\Omega_k} \langle (t_i^k - x_i) | \nabla u_i \rangle (V_i(t_i^k) - V_i(x_i)) e^{u_i} dy.
\end{aligned}$$

For A_1 and A_2 we use the fact that in Ω_k we have:

$$u_i(x) + 2 \log |x - t_i^k| \leq C,$$

to conclude that for $0 < s \leq 1$:

$$A_1 = A_2 = o(1),$$

we have integrals of the form:

$$A'_1 = \int_{\Omega_k} |\nabla u_i| e^{(1/2-s/2)u_i} dy = o(1),$$

and,

$$A'_2 = \int_{\Omega_k} |\nabla u_i| e^{(1/2-s/4)u_i} dy = o(1).$$

For A_3 we use the previous fact and the sup + inf inequality to conclude that for $1/2 < s \leq 1$:

$$A_3 = o(1)$$

because we have an integral of the form:

$$A'_3 = \int_{\Omega_k} |\nabla u_i| e^{(3/4-s/2)u_i} dy = o(1).$$

For A_4 we use integration by part to have:

$$A_4 = \int_{\partial\Omega_k} \langle (t_i^k - x_i) | \nu \rangle (V_i(t_i^k) - V_i(x_i)) e^{u_i} dy.$$

But, the boundary of Ω_k is the union of parts of mediatrices of segments linked to t_i^k . Let's consider a point t_i^j linked to t_i^k and denote $D_{i,j,k}$ the mediatrice of the segment (t_i^j, t_i^k) , which is in the boundary of Ω_k . Note that this mediatrice is in the boundary of Ω_j and the same decomposition for Ω_j gives us the following term:

$$A'_4 = - \int_{D_{i,j,k}} \langle (t_i^j - x_i) | \nu \rangle (V_i(t_i^j) - V_i(x_i)) e^{u_i} dy.$$

Thus, we have to estimate the sum of the 2 following terms:

$$A_5 = \int_{D_{i,j,k}} \langle (t_i^k - x_i) | \nu \rangle (V_i(t_i^k) - V_i(x_i)) e^{u_i} dy.$$

and,

$$A_6 = A'_4 = - \int_{D_{i,j,k}} \langle (t_i^j - x_i) | \nu \rangle (V_i(t_i^j) - V_i(x_i)) e^{u_i} dy.$$

We can write them as follows:

$$A_5 = \int_{D_{i,j,k}} \langle (x - x_i) | \nu \rangle (V_i(t_i^k) - V_i(x_i)) e^{u_i} dy + \int_{D_{i,j,k}} \langle (t_i^k - x) | \nu \rangle (V_i(t_i^k) - V_i(x_i)) e^{u_i} dy.$$

and,

$$A_6 = - \int_{D_{i,j,k}} \langle (x - x_i) | \nu \rangle (V_i(t_i^j) - V_i(x_i)) e^{u_i} dy - \int_{D_{i,j,k}} \langle (t_i^j - x) | \nu \rangle (V_i(t_i^j) - V_i(x_i)) e^{u_i} dy.$$

We can write:

$$\begin{aligned} & \int_{D_{i,j,k}} \langle (x - x_i) | \nu \rangle (V_i(t_i^k) - V_i(x_i)) e^{u_i} dy - \int_{D_{i,j,k}} \langle (x - x_i) | \nu \rangle (V_i(t_i^j) - V_i(x_i)) e^{u_i} dy = \\ & = \int_{D_{i,j,k}} \langle (x - x_i) | \nu \rangle (V_i(t_i^k) - V_i(t_i^j)) e^{u_i} dy = o(1), \end{aligned}$$

for $1/2 < s \leq 1$. Because, we do integration on the mediatrice of (t_i^j, t_i^k) , $|x - t_i^j| = |x - t_i^k|$, and:

$$|V_i(t_i^k) - V_i(t_i^j)| \leq 2A|x - t_i^k|^s$$

$$u_i(x) + 2 \log |x - t_i^k| \leq C,$$

and,

$$|x - x_i| \leq \delta_i \epsilon,$$

To estimate the integral of the following term:

$$e^{(3/4-s/2)u_i} \leq C r^{(-3/2+s)},$$

which is integrable and tends to 0, for $1/2 < s \leq 1$, because we are on the ball $B(x_i, \delta_i \epsilon)$.

In other part, for the term:

$$\int_{D_{i,j,k}} \langle (t_i^k - x) | \nu \rangle (V_i(t_i^k) - V_i(x_i)) e^{u_i} dy - \int_{D_{i,j,k}} \langle (t_i^j - x) | \nu \rangle (V_i(t_i^j) - V_i(x_i)) e^{u_i} dy.$$

We use the fact that, on $D_{i,j,k}$:

$$|x - t_i^j| = |x - t_i^k|,$$

$$u_i(x) + 2 \log |x - t_i^k| \leq C,$$

$$|V_i(t_i^k) - V_i(x_i)| \leq 2A|x_i - t_i^k|^s \leq \delta_i^s,$$

and,

$$|V_i(t_i^j) - V_i(x_i)| \leq 2A|x_i - t_i^j|^s \leq \delta_i^s,$$

To estimate the integral of the following term:

$$e^{(1/2-s/4)u_i} \leq C r^{(-1+s/2)},$$

which is integrable and tends to 0, because we are on the ball $B(x_i, \delta_i \epsilon)$.

Thus,

$$B_i = o(1),$$

Proof of the theorem 2:

Next, we use the formulation of the case of three blow-up points, see [2]. Because the blow-ups points are simple, we can consider the following function:

$$v_i(\theta) = u_i(x_i + r_i\theta) - u_i(x_i),$$

where r_i is such that:

$$r_i = e^{-u_i(x_i)/2},$$

$$\int_{B^\epsilon} V_i(x_i + \delta_i y) e^{v_i} \rightarrow 8\pi.$$

$$\begin{aligned} u_i(x_i + r_i\theta) &= \int_{\Omega} G(x_i + r_i\theta, y) V_i(y) e^{u_i(y)} dy = \\ &= \int_{\Omega - B(x_i, 2\delta_i\epsilon')} G(x_i, y) V_i e^{u_i(y)} dy + \int_{B(x_i, 2\delta_i\epsilon')} G(x_i + r_i\theta, y) V_i e^{u_i(y)} dy = \end{aligned}$$

We write, $y = x_i + r_i\tilde{\theta}$, with $|\tilde{\theta}| \leq 2\frac{\delta_i}{r_i}\epsilon'$,

$$\begin{aligned} u_i(x_i + r_i\theta) &= \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon')} \frac{1}{2\pi} \log \frac{|1 - (\bar{x}_i + r_i\tilde{\theta})(x_i + r_i\tilde{\theta})|}{r_i|\theta - \tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy + \\ &\quad + \int_{\Omega - B(x_i, 2\delta_i\epsilon')} G(x_i + r_i\theta, y) V_i e^{u_i(y)} dy \\ u_i(x_i) &= \int_{\Omega - B(x_i, 2\delta_i\epsilon')} G(x_i, y) V_i e^{u_i(y)} dy + \int_{B(x_i, 2\delta_i\epsilon')} G(x_i, y) V_i e^{u_i(y)} dy \end{aligned}$$

Hence,

$$\begin{aligned} u_i(x_i) &= \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon')} \frac{1}{2\pi} \log \frac{|1 - \bar{x}_i(x_i + r_i\tilde{\theta})|}{r_i|\tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy + \\ &\quad + \int_{\Omega - B(x_i, 2\delta_i\epsilon')} G(x_i, y) V_i e^{u_i(y)} dy \end{aligned}$$

We look to the difference,

$$v_i(\theta) = u_i(x_i + r_i\theta) - u_i(x_i) = \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon')} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy + h_1 + h_2,$$

where,

$$h_1(\theta) = \int_{\Omega - B(x_i, 2\delta_i\epsilon')} G(x_i + r_i\theta, y) V_i e^{u_i(y)} dy - \int_{\Omega - B(x_i, 2\delta_i\epsilon')} G(x_i, y) V_i e^{u_i(y)} dy,$$

and,

$$h_2(\theta) = \int_{B(0, 2\delta_i\epsilon')} \frac{1}{2\pi} \log \frac{|1 - (\bar{x}_i + r_i\tilde{\theta})y|}{|1 - \bar{x}_i y|} V_i e^{u_i(y)} dy.$$

Remark that, h_1 and h_2 are two harmonic functions, uniformly bounded.

According to the maximum principle, the harmonic function $G(x_i + r_i\theta, \cdot)$ on $\Omega - B(x_i, 2\delta_i\epsilon')$ take its maximum on the boundary of $B(x_i, 2\delta_i\epsilon')$, we can compute this maximum:

$$G(x_i + r_i\theta, y_i) = \frac{1}{2\pi} \log \frac{|1 - (\bar{x}_i + r_i\tilde{\theta})y_i|}{|x_i + r_i\theta - y_i|} \simeq \frac{1}{2\pi} \log \frac{(|1 + |x_i||\delta_i - \delta_i(3\epsilon' + o(1)))}{\delta_i\epsilon'} \leq C_{\epsilon'} < +\infty$$

with $y_i = x_i + 2\delta_i\theta_i\epsilon'$, $|\theta_i| = 1$, and $|r_i\theta| \leq \delta_i\epsilon'$.

We can remark, for $|\theta| \leq \frac{\delta_i\epsilon'}{r_i}$, that v_i is such that:

$$v_i = h_1 + h_2 + \int_{B(0, 2\frac{\delta_i\epsilon'}{r_i})} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy,$$

$$v_i = h_1 + h_2 + \int_{B(0, 2\frac{\delta_i\epsilon'}{r_i})} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta},$$

with h_1 and h_2 , the two uniformly bounded harmonic functions.

Remark: In the case of 2 or 3 or 4 blow-up points, and if we consider the half ball, we have supplementary terms, around the 2 other blow-up terms. Note that the Green function of the half ball is quasi-similar to the one of the unit ball and our computations are the same if we consider the half ball.

By the asymptotic estimates of Cheng-Lin, we can see that, we have the following uniform estimates at infinity. We have, after considering the half ball and its Green function, the following estimates:

$$\forall \epsilon > 0, \epsilon' > 0 \exists k_{\epsilon, \epsilon'} \in \mathbb{R}_+, i_{\epsilon, \epsilon'} \in \mathbb{N} \text{ and } C_{\epsilon, \epsilon'} > 0, \text{ such that, for } i \geq i_{\epsilon, \epsilon'} \text{ and } k_{\epsilon, \epsilon'} \leq |\theta| \leq \frac{\delta_i\epsilon'}{r_i},$$

$$(-4 - \epsilon) \log |\theta| - C_{\epsilon, \epsilon'} \leq v_i(\theta) \leq (-4 + \epsilon) \log |\theta| + C_{\epsilon, \epsilon'},$$

and,

$$\begin{aligned} \partial_j v_i &\simeq \partial_j u_0(\theta) \pm \frac{\epsilon}{|\theta|} + C \left(\frac{r_i}{\delta_i} \right)^2 |\theta| + m \times \left(\frac{r_i}{\delta_i} \right) + \\ &+ \sum_{k=2}^m C_1 \left(\frac{r_i}{d(x_i, x_i^k)} \right), \end{aligned}$$

In the case, we have:

$$\frac{d(x_i, x_i^k)}{\delta_i} \rightarrow +\infty \text{ for } k = 2 \dots m,$$

We have after using the previous term of the Pohozaev identity, for $1/2 < s \leq 1$:

$$\begin{aligned} o(1) &= J'_i = m' + \sum_{k=1}^m C_k o(1), \\ 0 &= \lim_{\epsilon'} \lim_{\epsilon} \lim_i J'_i = m', \end{aligned}$$

which contradict the fact that $m' > 0$.

here,

$$J_i = B_i = \int_{B(x_i, \delta_i\epsilon')} < x_1^i | \nabla(u_i - u) > (V_i - V_i(x_i)) e^{u_i} dy.$$

We use the previous formulation around each blow-up point.

If, for x_i^j , we have:

$$\frac{d(x_i^j, x_i^k)}{\delta_i^j} \rightarrow +\infty \text{ for } k \neq j, k = 1 \dots m,$$

We use the previous formulation around this blow-up point. We consider the following quantity:

$$J_i^j = B_i^j = \int_{B(x_i^j, \delta_i^j\epsilon')} < x_1^{i,j} | \nabla(u_i - u) > (V_i - V_i(x_i^j)) e^{u_i} dy.$$

with,

$$x_1^{i,j} = (\delta_i^j, 0),$$

In this case, we set:

$$v_i^j(\theta) = u_i(x_i^j + r_i^j \theta) - u_i(x_i^j),$$

where r_i^j is such that:

$$r_i^j = e^{-u_i(x_i^j)/2},$$

$$\int_{B(x_i^j, \delta_i^j \epsilon')} V_i(x_i^j + \delta_i^j y) e^{v_i} \rightarrow 8\pi.$$

We have, after considering the half ball and its Green function, the following estimates:

$$\forall \epsilon > 0, \epsilon' > 0 \exists k_{\epsilon, \epsilon'} \in \mathbb{R}_+, i_{\epsilon, \epsilon'} \in \mathbb{N} \text{ and } C_{\epsilon, \epsilon'} > 0, \text{ such that, for } i \geq i_{\epsilon, \epsilon'} \text{ and } k_{\epsilon, \epsilon'} \leq |\theta| \leq \frac{\delta_i^j \epsilon'}{r_i^j},$$

$$(-4 - \epsilon) \log |\theta| - C_{\epsilon, \epsilon'} \leq v_i^j(\theta) \leq (-4 + \epsilon) \log |\theta| + C_{\epsilon, \epsilon'},$$

and,

$$\begin{aligned} \partial_k v_i^j &\simeq \partial_k u_0^j(\theta) \pm \frac{\epsilon}{|\theta|} + C \left(\frac{r_i^j}{\delta_i^j} \right)^2 |\theta| + m \times \left(\frac{r_i^j}{\delta_i^j} \right) + \\ &+ \sum_{l \neq j}^m C_1 \left(\frac{r_i^j}{d(x_i^j, x_i^l)} \right), \end{aligned}$$

We have after using the previous term of the Pohozaev identity, for $1/2 < s \leq 1$:

$$o(1) = J_i^j = B_i^j = m' + \sum_{l \neq j}^m C_l o(1),$$

$$0 = \lim_{\epsilon'} \lim_{\epsilon} \lim_i J_i^j = m',$$

which contradict the fact that $m' > 0$.

If, for x_i^j , we have:

$$\frac{d(x_i^j, x_i^k)}{\delta_i^j} \leq C_{j,k} \text{ for some } k = k_j \neq j, 1 \leq k \leq m,$$

All the distances $d(x_i^j, x_i^k)$ are comparable with some δ_i^j . This means that we can use the Pohozaev identity directly. We can do this for example, for 4 blow-ups points.

We have many cases:

Case 1: the blow-up points are "equivalents", it seems that we have the same radius for the blow-up points.

Case 2: 3 points are "equivalents" and another blow-up point linked to the 3 blow-up points. We apply the Pohozaev identity directly with central point which link the 3 blow-up to the last.

Case 3: 2 pair of blow-up points separated.

Case 3.1: the 2 pair are linked: we apply the Pohozaev identity.

Case 3.2: the two pair are separated. It is the case of two separated blow-up points, see [1]

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